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Geodetic Graphs of Diameter Two and Some Related Structures*

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We show a connection between 2-connected geodetic graphs of diameter two and certain geometric structures; we apply this result, proving that these graphs fall into three classes: (1) strongly regular graphs with $\mu = 1$; (2) pyramids; (3) π-graphs. (The definitions for (1), (2), and (3) are given in the text.) We also give some results on graphs of type (3), including some non-existence conditions. Finally, we consider a more general class of graphs, showing a connection between some of them and Sperner spaces.

INTRODUCTION

A (finite, simple, undirected) graph is said to be *geodetic* if every pair of its vertices is connected by one and only one shortest path. Recently many authors have worked on geodetic graphs (see, e.g., [2, 7, 10, 11, 12]).

In the first section of this paper we give constructions and results on some classes of graphs, which we will apply in what follows. Next we present a classification of geodetic 2-connected graphs of diameter two. We also show (section 2) how geodetic graphs of diameter two with certain parameters are obtained from affine planes.

Finally, in Section 3 we deal with a larger class of graphs (F-geodetic), which are a natural generalization of both geodetic and distance-regular graphs; among other things, it is proved that there is a connection between some of these graphs and Sperner spaces.

1. GEODETIC GRAPHS OF DIAMETER TWO

1. In this paper we follow, when it is not otherwise specified, the terminology of [9]; we also suppose all sets in this paper to be *finite*.

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If G is a graph, then $V(G)$ ($E(G)$) is the vertex set (the edge set) of G. We denote by $(x; y)$ the unordered pair (edge) which consists of x and y.

A *clique* of a graph G is a maximal complete subgraph of G, with at least three vertices; a *clique-vertex* is a vertex which belongs to some clique.

Let G be a graph, let $x, y \in V(G)$. We denote by $d_G(x, y)$, or $d(x, y)$, the distance between x and y; moreover, we put

$$S^-(x, y) = \{z \in V(G) | (x; z) \in E(G), d(y, z) = d(x, y) - 1\},$$

$$S^+(x, y) = \{z \in V(G) | (x; z) \in E(G), d(y, z) = d(x, y) + 1\}.$$

A *distance-regular* graph is a connected graph G such that there are non-negative integers b_0, $b_1, ..., b_h$, c_0, $c_1, ..., c_h$ (with $c_0 = b_h = 0$ and $c_1 = 1$), where h is the diameter of G, such that for $x, y \in V(G)$ with $d(x, y) = j$ it is $|S^-(x, y)| = c_j$, $|S^+(x, y)| = b_j$ (see, e.g., [1]).

A *strongly regular graph* is a regular graph G for which there are non-negative integers λ and μ, such that the number of common neighbours of two different vertices is λ or μ, accordingly to whether they are or are not adjacent. It is known that these are exactly the distance-regular graphs of diameter two.

For a graph G, a shortest path between two vertices is, as in [3, 13], called a *geodesic*.

Let G be a connected graph of diameter h, and let F be a map from $\{1, 2, ..., h\}$ to the set of positive integers. We say that G is F-geodetic if, for every $i \in \{1, 2, ..., h\}$ and for every $x, y \in V(G)$, with $d(x, y) = i$, the number of geodesics between x and y is exactly $F(i)$.

Geodetic graphs are, therefore, F-geodetic graphs for which the map F is the constant 1; we call *proper* an F-geodetic but not a geodetic graph.

Let H be a graph of diameter $h \neq 1$, such that, for every $x \in V(H)$, there is a $y \in V(H)$ with $d(x, y) = h$. If there is a surjection φ: $V(H) \to A$ (where A is a set, disjoint from $V(H)$, such that $\varphi(x) = \varphi(y)$ if and only if $d(x, y) = h$ or 0, then H is said, with [4], to be *antipodal*. The map φ is said to be a *natural coloring* of H.

We call *fully antipodal* an antipodal graph H such that for $x, y \in V(H)$ with $\varphi(x) = \varphi(y)$ there is a $y' \in V(H)$ with $\varphi(y) = \varphi(y')$ and $(x, y') \in E(H)$. We call *perfectly anitpodal* a fully antipodal graph H such that for every $x, y \in V(H)$ with $\varphi(x) = \varphi(y)$ there is a $z \in V(H)$ with $(x; z)$, $(z; y) \in E(H)$. Such graphs are characterized by Theorem 2.

For the reader's convenience, we recall below a list of results shown in [13].

In what follows, *G is a 2-connected geodetic graph of diameter two.*

1. *If G is regular*, it has no cliques, or all vertices are clique-vertices [13, Theorems 5.3 and 5.6].

2. *If G is not regular* there are two integers m, n, with $1<m<n$, such that any vertex of G has degree m or n: clique-vertices and vertices whose distance from each clique is 2 have degree n, and the others have degree m [13, Theorem 5.6]; the number of vertices of G is $mn+1$ [13, Theorem 5.2]; all cliques of G have $k=n-m+2$ vertices [13, Theorem 5.5]; two vertices of degree m cannot be adjacent [13, Theorems 5.5 and 5.6].

3. If two clique-vertices are adjacent, they lie in the same clique [13, Lemma 5.7].

From now on we will use these results without mention; moreover, *if m, n are integers, with $1<m<n$, we denote by $\mathscr{G}(m, n)$ the class of all geodetic graphs as in result 2 above.*

2. We now introduce some definitions.

Let P and R be disjoint sets, let $I \subseteq P \times R$ be a relation, let $\| \subseteq R \times R$ be a reflexive and symmetric relation; we call *points* the elements of P, *lines* the elements of R, *incidence* the relation I, *parallelism* the relation $\|$.

We say that the ordered quadruple $(P, R, I, \|)$ is a *π-space* if the following axioms hold:

(A1) Two different points are incident with one and only one line.

(A2) For every point x and every line r, there is a unique line parallel to r and incident with x.

(A3) If two lines are neither parallel nor incident with the same point, there is a unique line parallel to both.

(A4) There are at least two points, which are not incident with the same line; every line is parallel to at least two other lines, parallel to each other.

Moreover, we say that $(P, R, I, \|)$ is a *Sperner space* (see, e.g., [8]) if it verifies (A1), (A2), and (A4), and there is a nonnegative integer k such that each line is incident with exactly k points, and $\|$ is transitive.

If $(P, R, I, \|)$ is an affine plane (of order $q \geqslant 3$), it is of course a π-space; moreover, let D be the set of equivalence classes of $\|$; put $R' = R \cup D$, $\|' = \| \cup \{(r; r') | r \in r',\ r \in R,\ r' \in D\}$. Then $(P, R', I, \|')$ is again a π-space, which we call a *modified affine plane.*

We state the following without proof:

Let G be a 2-connected, non-regular, geodetic graph of diameter 2, without vertices at distance 2 from each clique. Let R be the set of all clique-vertices of G; put $P = V(G) \backslash R$, $I = \{(x, r) \in P \times R | (x; r) \in E(G)\}$, $\| = \{(r, r') \in R \times R | (r; r') \in E(G)$ or $r = r'\}$. Then $(P, R, I, \|)$ is a π-space.

Conversely, if $(P, R, I, \|)$ *is a* π*-space, by* $V(G) = P \cup R$ *and* $E(G) = \{(x; r) | (x, r) \in I\} \cup \{(r; r') | (r, r') \in \|$ *and* $r \neq r'\}$, *we obtain a* 2*-connected, non-regular, geodetic graph of diameter* 2, *without vertices at distance* 2 *from each clique.*

DEFINITION I. A π-graph is a graph which satisfies the above conditions.

We now introduce the following notation, *which is used throughout this paper.*

Let $\mathscr{P} = (P, R, I)$ *be a projective plane of order* $q \geqslant 3$, and let σ be a *polarity of* $\mathscr{P}$. A *point* x of $\mathscr{P}$ is said to be *absolute* for σ if $\sigma(x) = x$ (see [5]). *We denote by* $G(\mathscr{P}, \sigma)$ the graph G defined by $V(G) = P$ and $E(G) = \{(x; y) | xI\sigma(y)\}$. Moreover, if the *absolute points* of σ are exactly the points incident with a *line* r, *we denote by* $H(\mathscr{P}, \sigma)$ *the section graph of* G *defined by the set of non-absolute points different from* $\sigma(r)$.

The graph $G(\mathscr{P}, \sigma)$ (the graph $H(\mathscr{P}, \sigma)$ will be called *projective* (*restricted projective*).

LEMMA 1. *Let* H *be a fully antipodal graph, and let* φ: $V(H) \to A$ *be a natural coloring of* H. *Then* H *is complete multipartite or it is regular of degree* $|A| - 1$ *and diameter* 3.

Proof. Let h be the *diameter* of H: *we show that* $h \leqslant 3$. By contradiction, we suppose $h > 3$.

If $b, b' \in V(H)$, with $d(b, b') = 2$, then $\varphi(b) \neq \varphi(b')$, so by definition there is a z with $(z; b') \in E(H)$ and $\varphi(z) = \varphi(b)$. But now z and b have the same color and their distance is less than the diameter, contradicting the definition of antipodal graph. *Then* $h \leqslant 3$ and, by definition again, $h \neq 1$.

If $h = 2$, by [4, Corollary 3.6] the graph H is complete multipartite.

Consider now the remaining case $h = 3$. Since each vertex b of H is adjacent to at least one vertex of every color different from $\varphi(b)$, the degree of b is at least $|A| - 1$; since two vertices of the same color must have distance 3, the degree of b is exactly $|A| - 1$. The lemma is now proved.

EXAMPLE. The graph illustrated by Fig. 1 (colored with 1, 2, 3, 4) is fully antipodal of degree 3 and diameter 3, and also F-geodetic (with $F(2) = 1$, $F(3) = 2$). It is not distance-regular, and of course not perfectly antipodal.

THEOREM 2. *A graph is perfectly antipodal if and only if it is a restricted projective graph, or a complete multipartite graph with at least three parts.*

Proof. By Lemma 1, we must only show that a graph of diameter 3 is perfectly antipodal if and only if it is a restricted projective graph.

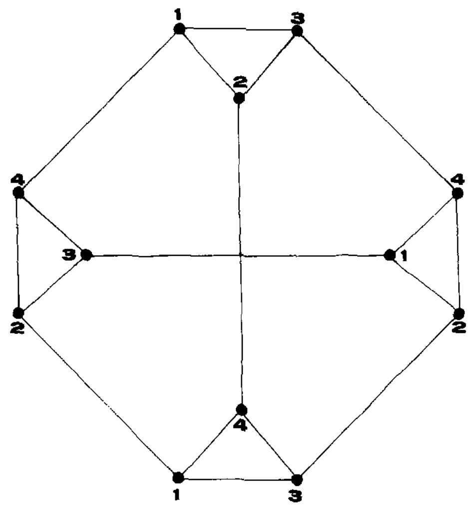

FIGURE 1

Let H be a perfectly antipodal graph of diameter 3, φ: $V(H) \to A$ be a natural coloring of H: then H has diameter 3 and it is regular of degree $q = |A| - 1$ (Lemma 1).

Let $c \notin A \cup V(H)$, $P = V(H) \cup A \cup \{c\}$. For $x \in P$ we state:

$$\sigma(x) = \begin{cases} \{ y \in V(H) | (x; y) \in E(H)\} \cup \{\varphi(x)\} & \text{if } x \in V(H) \\ \{ y \in V(H) | \varphi(y) = x\} \cup \{x, c\} & \text{if } x \in A \\ A & \text{if } x = c. \end{cases}$$

Finally, we put $R = \{\sigma(x) | x \in P\}$. Now we consider the incidence structure $\mathscr{P} = (P, R, \in)$. It is easy to show that two points are incident with at least one line.

We show that every line of $\mathscr{P}$ is incident with exactly $q + 1$ points. First, it is clear that it is true for all lines $\sigma(x)$, with $x \in V(H) \cup \{c\}$.

For $a \in A$, consider the line $\sigma(a)$. Let b be a vertex of H: every line for b must intersect $\sigma(a)$. Let $\sigma(x)$, $\sigma(y)$ be twe lines incident with b and intersecting $\sigma(a)$ in the same point a'. Then $\varphi(x) = \varphi(y) = a'$; since b is adjacent to both x and y, we have $d(x, y) \neq 3$. The hypotheses on H thus imply $x = y$. Then every line by b intersects $\sigma(a)$ in a unique point, so the points of $\sigma(a)$ are in one-to-one correspondence with the lines for b, thus are $q + 1$.

It is now easy to see that $\mathscr{P}$ is a projective plane of order q and that σ is a polarity of $\mathscr{P}$, whose absolute points are the elements of A, and $H = H(\mathscr{P}, \sigma)$.

Conversely, let $H = H(\mathscr{P}, \sigma)$ be a restricted projective graph, let A be the set of all absolute points of σ, let c be the pole of the line incident with all the points of A, and let φ: $V(H) \to A$ be the map which maps $b \in V(H)$ in the intersection of A with the line for b and c.

We show that H has diameter 3 *and is antipodal.* Let $b, b' \in V(H)$, with $b \neq b'$, such that b, b', c lie in the same line (note that in the definitions of projective and restricted projective graph we suppose that $q \geqslant 3$: so in a line for c there are at least two points not in A). Since their polars must intersect in a point of the polar of c, i.e., of A, in H no vertex is adjacent to both: so $d(b, b') \geqslant 3$.

With similar arguments it is now easy to conclude the proof.

DEFINITION II. A graph G is a pyramid of type h if there is a graph H of diameter h and there is a map φ from $V(H)$ to a set A (disjoint from $V(H)$), such that:

(1) the graph H is perfectly antipodal and φ is a natural coloring of H, or H is a complete graph and φ is a bijection;

(2) there is a $c \in V(G)$ such that $V(G) = V(H) \cup A \cup \{c\}$ and $E(G) = E(H) \cup \{(b; \varphi(b) | b \in V(H)\} \cup \{(a; c) | a \in A\}$.

THEOREM 3. *Let G be a 2-connected geodetic graph of diameter* 2. *If c is a vertex of G, whose distance from all cliques is two, then G is a pyramid of type* 1 *or* 3.

Proof. Let $A = \{x \in V(G) | (x; c) \in E(G)\}$, $B = V(G) \backslash (A \cup \{c\})$; let H be the section graph of G defined by B.

Since the elements of B are at distance 2 from c, for every $b \in B$ there is an $a \in A$ adjacent to b, which must be unique (because if it is not so, c and b are in a circuit of length 4, thus are adjacent). Now we put $\varphi(b) = a$, obtaining a map from B to A.

If H is a complete graph, then the map φ is injective: thus, for $b \in B$, $\varphi(b)$ has degree 2, so [13, Theorem 5.7 and Corollary 4] G is a pyramid of Type 1.

If H is not complete, let $b, b' \in B$ with $b \neq b'$ and $\varphi(b) = \varphi(b')$. If $(b; b') \in E(H)$, we have $(b; b') \in E(G)$, so b, b' and $\varphi(b)$ are in the same clique; the same holds if $d_H(b, b') = 2$. But this is a contradiction, since $\varphi(b)$ is not a clique-vertex: thus $d_H(b, b')$ is greater than 2.

Let x be an element of B, adjacent to b; the vertex y such that (x, y, b') is a geodesic cannot be $\varphi(b)$, so $y \in B$. This proves that $d_H(b, b') = 3$.

Conversely, if $b, b' \in B$ have distance 3 in H, from the fact that G has diameter 2 it follows that $\varphi(b) = \varphi(b')$. *Thus H is antipodal and φ is a natural coloring of it.*

It is now easy to conclude the proof.

COROLLARY 4. *The 2-connected geodetic graphs of diameter two fall in the following classes*:

(1) *strongly regular graphs with* $\mu = 1$;

(2) *pyramids of type* 1 *or* 3;

(3) π-*graphs.*

2. On π-Graphs

3. We call *principal* a line of a π-space which intersects every line which is not parallel to it. This definition is analogous to that given in [8] for weak affine spaces.

LEMMA 5. *Let S be a π-space, without lines incident with no points, and let G be the π-graph associated to S.*

If $G \in \mathscr{G}(m, n)$, *the following hold*:

(1) *a line is principal iff it has exactly* $m-1$ *points*;

(2) *if a line is principal, all lines parallel to it are also principal*;

(3) *if there are two principal lines which are not parallel, then S is an affine plane.*

Proof. If the line r has $m-1$ points, let s be a line which does not intersect r. Counting the lines by a point x of s it is clear that s is parallel to r, and r is principal; the other part of (1) is trivial. Thus (1) follows.

Let r be a principal line. By (1) the number of lines parallel to r is $n-m+1=k-1$. Therefore r is in an unique clique.

We suppose, by contradiction, that $s \| r$ is *not* principal; let s' be a line of S, such that s and s' are neither parallel nor incident with the same point of S. Let s'' be the parallel to s and s'.

Since r is principal it follows that $s'' \| r$, and similarly $s' \| r$: then $s \| s'$, contradicting the hypothesis on s'. Thus (2) follows.

Now (3) follows easily from (1) and (2).

The next theorem has been proved, in a different way, in [6, Lemma 3.6].

THEOREM 6. *Let* $q \geqslant 4$ *be an integer. A graph G is in* $\mathscr{G}(q+1, 2q-1)$ *if and only if G is a π-graph, whose π-space is an affine plane of order q.*

Before the proof, we remark that if m, n are integers, with $2 < m < n-1$, the class $\mathscr{G}(m, n)$ consists only of π-graphs; on the other hand in pyramids of type 1 it is $m=2$, and in pyramids of type 3 it is $m=n-1$ (since such a pyramid is a projective graph; see Theorem 2).

Proof. Let $G \in \mathscr{G}(q+1, 2q-1)$. Since q is greater than 3, the above remark shows that *G is a π-graph.* Let $S = (P, R, I, \|)$ be the associate π-space.

If two elements of R are adjacent in G, then they lie in the same clique, so the number of elements of R adjacent to a given $r \in R$ is a multiple of $k-1$ (where k is the number of the vertices of a clique).

In our case it is $m = q+1$, $n = 2q-1$, so $k-1 = q-1$. Thus r has $q-1$ or $2(q-1)$ neighbours in R, i.e., respectively, it has q or one neighbours in P.

Let x, y, and z be three different points not in a line. The lines by x and y and by x and z have at least two points, then they have q points: because $q = m-1$, S is an affine plane (Lemma 5).

Since the converse is straightforward, the assertion is proved.

THEOREM 7. *Let $q \geqslant 4$ be an integer. A graph G is in $\mathscr{G}(q+1, 2q)$ if and only if G is a π-graph, whose π-space is a modified affine plane of order q.*

Proof. Let $G \in \mathscr{G}(q+1, 2q)$. By the hypothesis on q and the remark before the proof of Theorem 6, it is clear that G is a π-graph: let $S = (P, R, I, \|)$ be its π-space, let R' be the set of all lines incident with some point, let $H = R \backslash R'$, and let $\|'$ be the restriction of $\|$ to R'; *we prove that $S' = (P, R', I, \|')$ is an affine plane.*

Given $r \in R$, since the degree of r in G is $2q$ and since r is in at least one clique, from $k = 2q - (q+1) + 2 = q+1$ (where k is the number of elements of a clique) it follows that r is adjacent to q element of P or to none: so r is incident to no points or to exactly q points. Then *every line of S' has q points.* Reasoning as in Theorem 6 we conclude that *S' is an affine plane.*

Since the converse is straightforward, the assertion follows.

4. The graphs considered in Theorems 6 and 7 do not exhaust the class of π-graphs: other examples are the projective graphs $G(\mathscr{P}, \sigma)$, where the absolute points of σ are not collinear.

LEMMA 8. *Let S be a π-space, whose π-graph is in $\mathscr{G}(m, n)$. If in S there are principal lines, one of the following holds:*

(1) *the π-space S is an affine plane of order $m-1$;*

(2) *each non-principal line of S is incident with exactly $k = n - m + 2$ points.*

Proof. If S is an affine plane of order q, we have $q = m-1$ (Theorem 6), so (1) holds.

Now we suppose that S is *not* an affine plane. Let r be a principal line of S; by Lemma 5 there are exactly k lines parallel to r. A non-principal line s

is not parallel to any of these lines, so it intersects each principal line; on the other hand, each point of s lies in some line parallel to r, hence in a principal line. Thus (2) holds and the lemma is proved.

Applying the preceding results, we prove

THEOREM 9. *Let m and n be positive integers, with $m \geqslant 5$ and $3m \leqslant 2n+2$. Then $\mathscr{G}(m, n)$ is non-empty if and only if n is equal to $2(m-1)$ or $2(m-1)-1$ and there is an affine plane of order $m-1$.*

Proof. The "if" part is an easy consequence of Theorems 6 and 7. Let us suppose that $\mathscr{G}(m, n)$ is non-empty.

Since by definition the elements of $\mathscr{G}(m, n)$ are non-regular graphs, in $\mathscr{G}(m, n)$ there are only pyramids and π-graphs. As observed before the proof of Theorem 9, if in $\mathscr{G}(m, n)$ there are pyramids, it is $(m, n)=(2, n)$ or $(m, n)=(m, m+1)$. Both cases contradict our hypothesis on n, m (in the second it is $2n+2=2m+4$, so $m \leqslant 4$). *Therefore, all elements of $\mathscr{G}(m, n)$ are π-graphs.*

Let $G \in \mathscr{G}(m, n)$, and let S be the π-space of G. It is easily seen that the points on a line r of S are $n-t(k-1)$, where t is the number of cliques of G containing r. For $t \geqslant 3$ we have $n-t(k-1) \leqslant n-3(k-1)= 3m-2n-3<0$, so a line of S is in at most two cliques.

A line of S has, therefore, $m-1$ or $2m-n-2$ points. *By contradiction*, we now suppose that no line has $m-1$ points.

Then S is a BIB-design (with $\lambda=1$); putting $v=|P|$, we have, therefore, $(v-mn-1)(2m-n-2)=vm$, $v-1=m(2m-n-3)$.

Performing some calculations, from the above equalities we have $2(m^2+n^2)+5(mn-1)+7(n-m)=0$, a contradiction since n is greater than m. Then, *there is a line incident with $m-1$ points.*

If in S no line is incident with no points and if S is not an affine plane, in S there are principal lines (Lemma 5), so $k=2m-n-2$ (Lemma 8), so $n-m+2=2m-n-2$, i.e., $2n+4=3m$, contradicting the hypothesis on m and n. Then *if no line is incident with no points, then S is an affine plane and $n=2(m-1)-1$.*

Finally, if there is a line incident with no points, then $2m-n-2=0$, i.e., $n=2(m-1)$: this proves that there is an affine plane of order $m-1$ (Theorem 7), completing the proof.

3. On F-Geodetic Graphs

5. A simple characterization of F-geodetic graphs is given by

THEOREM 10. *A graph G of diameter h is F-geodetic for some F if and only if there are positive integers $c_1=1, c_2, \ldots, c_h$ such that* $\operatorname{diam}(G)=h$ *and,* *for every* $j \in \{1, 2, \ldots, h\}$, *if* $d(x, y) = j$ *then* $|S^-(x, y)| = c_j$. *Moreover,* $F(j) = c_j \cdot F(j-1)$ *for each* $j \in \{2, 3, \ldots, h\}$.

Proof. Suppose that G is F-geodetic; let $j \in \{1, 2, \ldots, h\}$. For $x, y \in V(G)$ with $d(x, y) = j$, for every $z \in S^-(x, y)$ there are $F(j-1)$ geodesics between z and y; adding x to each of them we obtain the same number of geodesics between x and y.

Since every geodesic between x and y must intersect $S^-(x,)$ in a singleton, the geodesics between x and y are $|S^-(x,)y)| \cdot F(j-1)$ in all. Therefore $F(j) = |S^-(x, y)| \cdot F(j-1)$, i.e., $|S^-(x, y)| = F(j)/F(j-1)$. This proves that there are $c_1, c_2, \ldots, c_h$ with the requested property.

With a similar argument one can prove that if there are such integers, then G is F-geodetic. The last part of the assertion is now obvious.

PROPOSITION 11. *Let* $S = (P, R, I, \|)$ *be a Sperner space, in which each line is incident with* k *points; suppose that* S *is an affine plane. Put* $V(G) = P \cup R$ *and*

$$E(G) = \{(r; r') | r, r' \in R, r \neq r', r \| r'\} \cup \{(x; r) | x \in P, r \in R, xIr\};$$

then the graph G *has diameter* 3 *and it is F-geodetic, with* $F(2) = 1$, $F(3) = 2k$.

Proof. Let $a, b \in V(G)$; it is easy to see that:

—if $a, b \in P$, then $d(a, b) = 2$ and there is a unique geodesic between a and b;

—if $a \in P$, $b \in R$ (or $a \in R$, $b \in P$), then $d(a, b) \leqslant 2$ and there is a unique geodesic between a and b.

Now we suppose $a, b \in R$. If $a \| b$ it is clear that a and b are adjacent in G; if there is a point x, incident with both a and b, then (a, x, b) is the unique geodesic between them.

There remains the case in which a and b are neither parallel nor incident with the same point: but then $d(a, b) \geqslant 3$. Moreover, if x is incident to a, let c be the parallel to b by x, then (a, x, c, b) is clearly a geodesic: so $d(a, b) = 3$. For each y incident to b, let d be the parallel to a by y; then (a, d, y, b) is a geodesic.

It is easily seen that all geodesics between a and b are obtained in this way; since every line has k points, these are $2k$.

As S is not an affine plane, we conclude that G has diameter 3, and this completes the proof.

We call *minimally F-geodetic* an F-geodetic graph G such that, for every $B \subseteq V(G)$, if $B \neq V(G)$ then the section graph of G defined by B is not F-geodetic (with the *same* map F, hence with the same diameter).

THEOREM 12. *Every proper, minimally F-geodetic graph is 2-connected.*

Proof. Namely let G be a proper, minimally F-geodetic graph of diameter h; we suppose, by contradiction, that it is not 2-connected. Let c be a cut-vertex of G, and let $(A; B)$ be the corresponding decomposition of G.

Since every geodesic between vertices of A must lie in A, if there are $a, a' \in A$ with $d(a, a') = h$, the section graph of G defined by A is F-geodetic, a contradiction: therefore $d(a, a') < h$ for each $a, a' \in A$ and, similarly, $d(b, b') < h$ for each $b, b' \in B$.

Then there are $a' \in A$, $b' \in B$ such that $d(a', b') = h$: from now on we consider these vertices fixed.

Since G is not geodetic, let i be the *least* integer such that $F(i) \neq 1$. On a geodesic between a' and b' we choose $a \in A$, $b \in B$ with $d(a, b) = i$.

Every geodesic between a and b is the union of a geodesic from a to c and one from c to b. Since $d(a, c)$, $d(c, b)$ are less than i, by our hypothesis on i both geodesics are uniquely determined. There is now a unique geodesic between a and b, contradicting the assumption $F(i) \neq 1$. This completes the proof.

We conclude with the following

EXAMPLE. The graph G in Fig. 2 is F-geodetic of diameter 4, with $F(2) = F(3) = 1$ and $F(4) = 2$; it is neither 2-connected nor minimally F-geodetic (namely the section graph of G defined by $V(G) \setminus \{1\}$ is F-geodetic).

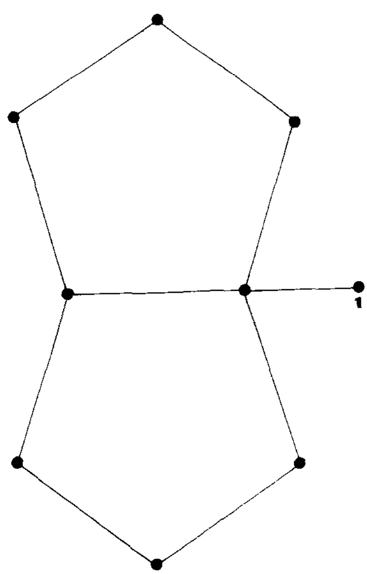

FIGURE 2

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